B. APPENDIX ON FIBRE BUNDLES

In this appendix, I give concise definitions of those concepts from the theory of fibre bundles which are necessary to the analogy between metric-connection theories of gravity and gauge theories of elementary particle interactions. In this brief account, no mention is made of continuity and differentiability. All spaces and maps are assumed to be as continuous and differentiable as necessary. For more details on the theory of fibre bundles, I refer the reader to the standard texts by Steenrod [1951] and by Kobayashi and Nomizu [1963 , 1969] and to my own lecture notes [1979].

A fibre bundle is a topological space, B, which consists of a copy, F_p, of a space, F, called the typical fibre, at each point, p, of a base space, M. As a set, the bundle space is the union

$$B = \bigcup_{p \in M} F_p , \tag{1}$$

and there is a bundle projection,

$$\pi : B \to M : \psi \to p, \qquad \text{if } \psi \in F_p. \tag{2}$$

The fibre over p is just $F_p = \pi^{-1}(p)$. The bundle space must have a topology which makes the fibre bundle locally trivial. This means that each point, $p \in M$, has a neighborhood, $U \subset M$, for which there is a homeomorphism,

$$h : \pi^{-1}(U) \to U \times F, \tag{3}$$

which preserves fibres; i.e. $\pi_1 \circ h = \pi$, where $\pi_1 : U \times F \to U$ is the projection on the first argument. Such a pair, (U,h), is called a triviality chart; U is the triviality patch and h is the triviality map. An atlas is a set of charts, $\{(U_\alpha, h_\alpha)\}$, whose patches cover the base space; i.e.

$$\bigcup_\alpha U_\alpha = M. \quad (4)$$

A local cross section, ψ, of a fibre bundle, B, defined over a subset, $U \subset M$, is a function, $\psi : U \to B$, such that $\pi \circ \psi = id_U$. Thus, to each point, $p \in U$, ψ assigns a point, $\psi(p)$, in the fibre over p. It is global if it is defined over all of M; i.e. $\psi : M \to B$.

The map

$$h_{\alpha p} = \pi_2 \circ h_\alpha \Big|_{\pi^{-1}(p)} : \pi^{-1}(p) \to F, \quad (5)$$

(where $\pi_2 : U \times F \to F$ is the projection on the second argument) is a homeomorphism from the fibre over p to the typical fibre. If $\psi \in \pi^{-1}(p)$, we sometimes write $\overset{\alpha}{\psi}$ for $h_{\alpha p}(\psi)$ and say that $\overset{\alpha}{\psi}$ is the description of ψ in the α-trivialization. The continuous function,

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to \text{Homeo}(F,F), \quad (6)$$

defined by

$$h_{\alpha\beta}(p) = h_{\alpha p} \circ h_{\beta p}^{-1} : F \to F, \quad (7)$$

relates the description of ψ in the α-trivialization to its description in the β-trivialization:

$$\overset{\alpha}{\psi} = h_{\alpha p} \circ h_{\beta p}^{-1} \overset{\beta}{\psi} = h_{\alpha\beta}(p) \overset{\beta}{\psi}. \quad (8)$$

Hence $h_{\alpha\beta}$ is called the transition function from the β-trivialization to the α-trivialization, and the homeomorphism, $h_{\alpha\beta}(p)$, is called the action of the transition function.

Now suppose there is a left action, ℓ, of a Lie group, G, on the typical fibre, F. A G-bundle is a fibre bundle, B, together with an atlas, $\{(U_\alpha, h_\alpha)\}$, and a set of functions,

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G, \tag{9}$$

called overlap gauge transformations, such that for all $p \in U_\alpha \cap U_\beta$, the action of the transition function, $h_{\alpha\beta}(p)$, coincides with the left action of the overlap gauge transformation,

$$\ell_{g_{\alpha\beta}(p)} = h_{\alpha\beta}(p) = h_{\alpha p} \circ h_{\beta p}^{-1} : F \to F, \tag{10}$$

and such that for all $p \in U_\alpha \cap U_\beta \cap U_\gamma$, the overlap gauge transformations satisfy the cocycle condition,

$$g_{\alpha\beta}(p)\, g_{\beta\gamma}(p) = g_{\alpha\gamma}(p), \tag{11}$$

where the product on the left is the group product. The group, G, is called the structure group or the gauge group. The pair (U_α, h_α) is now called a gauge chart; U_α is the gauge patch and h_α is the gauge map.

A G-vector bundle is a G-bundle, E, in which the typical fibre is a vector space, V, and the left action is a representation, R, in that it is linear. A principal G-bundle is a G-bundle, P, in which the typical fibre is the group, G, itself, and the left action, L, is left multiplication in the group,

$$L_{g_1}\, g_2 = g_1\, g_2. \tag{12}$$

Two G-bundles are called associated if they have the same base space, M, the same gauge group, G, the same gauge patches, $\{U_\alpha\}$, and most important, the same overlap gauge transformations, $\{g_{\alpha\beta}\}$.

There are many ways to define a connection on a G-bundle. I only give two of them. They are not the most geometric nor the most illuminating; rather they are the ones most useful in particle physics and relativity.

The first definition is only appropriate when G is a matrix group, although it applies to any G-bundle, B, with patches, $\{U_\alpha\}$, and overlap gauge transformations, $\{g_{\alpha\beta}\}$. A <u>base connection 1-form</u>, A, on a G-bundle, B, is a Lie algebra valued 1-form, $\overset{\alpha}{A}$, on each patch, U_α,

$$\overset{\alpha}{A} : TU_\alpha \to \mathcal{L}G, \tag{13}$$

($\mathcal{L}G$ is the Lie algebra of G.) such that on each overlap, $U_\alpha \cap U_\beta$, the 1-forms are related by

$$\overset{\alpha}{A} = g_{\alpha\beta} \overset{\beta}{A} g_{\alpha\beta}^{-1} + g_{\alpha\beta}\, d(g_{\alpha\beta}^{-1}), \tag{14}$$

where d takes the differential of each matrix element and all products are matrix multiplication. The 1-form, $\overset{\alpha}{A}$, is called the <u>connection 1-form in the α-gauge</u>.

(If G were not a matrix group, the definition would still hold except that equation (14) would have to be generalized. See Kobayashi and Nomizu [1963] pp. 65-66.)

Notice that a connection on one G-bundle is also a connection on all of its associated G-bundles.

The connection 1-form can be expanded in a coordinate basis, dx^a, for the 1-forms and a dimensionless basis, T_P, for the Lie algebra, $\mathcal{L}G$:

$$\overset{\alpha}{A} = \overset{\alpha}{A}{}^P{}_a\, T_P\, dx^a. \tag{15}$$

The components, $\overset{\alpha}{A}{}^{p}{}_{a}$, are called the connection coefficients in the α-gauge or more physically, the gauge potentials in the α-gauge.

Now consider a G-vector bundle, E, with typical fibre, V, representation, R, and overlap gauge transformations, $g_{\alpha\beta}$, which has a base connection 1-form, A. Under a gauge transformation, the components of a cross section, ψ, transform according to

$$\overset{\alpha}{\psi}{}^{k} = (Rg_{\alpha\beta})^{k}{}_{j}\,\overset{\beta}{\psi}{}^{j}. \tag{16}$$

Hence, the directional derivative of the components in the direction, $X = X^a\,\partial_a$, transform according to

$$X^a\,\partial_a\overset{\alpha}{\psi}{}^{k} = (Rg_{\alpha\beta})^{k}{}_{j}\,X^a\,\partial_a\overset{\beta}{\psi}{}^{j} + [X^a\,\partial_a(Rg_{\alpha\beta})^{k}{}_{j}]\overset{\beta}{\psi}{}^{j}. \tag{17}$$

The non-covariant second term prevents $X^a\,\partial_a\overset{\alpha}{\psi}{}^{k}$ from being the components of a new cross section. On the other hand, the combination,

$$\nabla_X\overset{\alpha}{\psi}{}^{k} = X^a(\partial_a\overset{\alpha}{\psi}{}^{k} + \overset{\alpha}{A}{}^{p}{}_{a}(RT_p)^{k}{}_{j}\,\overset{\alpha}{\psi}{}^{j}), \tag{18}$$

called the covariant derivative of ψ in the direction, X, is a cross section because it transforms covariantly:

$$\nabla_X\overset{\alpha}{\psi}{}^{k} = (Rg_{\alpha\beta})^{k}{}_{j}\,\nabla_X\overset{\beta}{\psi}{}^{j}. \tag{19}$$

This brings us to the second definition of a connection, in the form of a covariant derivative on a vector bundle. For the time being, ignore the definition (18) and let E be an arbitrary vector bundle (without a connection 1-form). A covariant derivative on a vector bundle, E, is a function, ∇, which to each local cross section, $\psi : U \to E$, and local tangent vector field, $X : U \to TU$, assigns a new local cross section, $\nabla_X\psi : U \to E$, (called the covariant derivative of the cross section, ψ, in the direction, X,) which is

(i) function linear in the differentiating direction,

$$\nabla_{(fX + gY)}\psi = f\,\nabla_X\psi + g\,\nabla_Y\psi; \tag{20}$$

(ii) additive in the differentiated cross section,

$$\nabla_X(\psi + \chi) = \nabla_X\psi + \nabla_X\chi\ ; \tag{21}$$

and

(iii) Leibnizian in the differentiated cross section,

$$\nabla_X(f\psi) = X^a(\partial_a f)\psi + f\ \nabla_X\psi. \tag{22}$$

(If E is not a vector bundle, then there is no definition of a covariant derivative.)

In the case that E is a G-vector bundle with representation, R, equation (18) defines a covariant derivative, ∇, in terms of a base connection 1-form, A. Conversely, if R is a faithful representation of $\mathcal{L}G$ (i.e. the matrices RT_p are linearly independent), then A may be recovered from ∇ by reading off the coefficients of RT_p in

$$(\nabla_a \overset{\alpha}{e}_j)^k = \overset{\alpha}{A}{}^p{}_a\ (RT_p)^k{}_j, \tag{23}$$

where $\overset{\alpha}{e}_j$ is the local frame field for the bundle, E, which has the components, $(\overset{\alpha}{e}_j)^k = \delta^k_j$, in the α-gauge. If R is *not* a faithful representation of $\mathcal{L}G$, then ∇ does *not* fully determine the base connection 1-form, A. In that case, A is regarded as the connection rather than ∇.

From the connection, one defines the curvature. In terms of the base connection 1-form, A, one defines the <u>base curvature 2-form</u>, F, which in the α-gauge is the $\mathcal{L}G$ valued 2-form,

$$\overset{\alpha}{F} = \overset{\alpha}{F}{}^p{}_{ab}\ T_p\ dx^a \wedge dx^b, \tag{24}$$

whose components (in a coordinate basis) are the gauge fields in the α-gauge,

$$\overset{\alpha}{F}{}^{P}{}_{ab} = \partial_a \overset{\alpha}{A}{}^{P}{}_{b} - \partial_b \overset{\alpha}{A}{}^{P}{}_{a} + f^{P}{}_{QR}\, \overset{\alpha}{A}{}^{Q}{}_{a}\, \overset{\alpha}{A}{}^{R}{}_{b}, \tag{25}$$

where the $f^{P}{}_{QR}$ are the dimensionless structure constants of $\mathcal{L}G$ defined by

$$[T_Q, T_R] = f^{P}{}_{QR}\, T_P. \tag{26}$$

In terms of the covariant derivative, ∇, one defines the curvature operator, F, by the formula,

$$F(X,Y)\psi = \nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi, \tag{27}$$

and finds that in the α-gauge

$$F(X,Y)\overset{\alpha}{\psi}{}^{k} = X^a\, Y^b\, \overset{\alpha}{F}{}^{P}{}_{ab}\, (RT_P)^{k}{}_{j}\, \overset{\alpha}{\psi}{}^{j}\ , \tag{28}$$

where $\overset{\alpha}{F}{}^{P}{}_{ab}$ is again given by (25). Under a gauge transformation the curvature transforms covariantly according to

$$\overset{\alpha}{F} = g_{\alpha\beta}\, \overset{\beta}{F}\, g^{-1}_{\alpha\beta}\ , \tag{29}$$

or

$$\overset{\alpha}{F}{}^{P}{}_{ab} = \mathrm{ad}(g_{\alpha\beta})^{P}{}_{Q}\ \overset{\beta}{F}{}^{Q}{}_{ab}, \tag{30}$$

where ad denotes the adjoint representation of G. Hence, the curvature may be regarded as a global cross section of the associated G-vector bundle whose typical fibre is the tensor product of the spacetime 2-forms and the Lie algebra of G transforming under the adjoint representation.